

ON WEAK r -HELIX SUBMANIFOLDS

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ABSTRACT. In this paper, we investigate special curves on a weak r -helix submanifold in Euclidean n -space E^n . Also, we give the important relations between weak r -helix submanifolds and the special curves such as line of curvature, asymptotic curve and helix line.

1. INTRODUCTION

In differential geometry of manifolds, an helix submanifold of \mathbb{R}^n with respect to a fixed direction d in \mathbb{R}^n is defined by the property that tangent planes make a constant angle with the fixed direction d (helix direction) in [4]. Di Scala and Ruiz-Hernández have introduced the concept of these manifolds in [4]. Besides, the concept of weak r -helix submanifold of \mathbb{R}^n was introduced in [3]. Let $M \subset \mathbb{R}^n$ be a submanifold. We say that M is a weak r -helix if there exist r linearly independent directions d_1, \dots, d_r , such that M is a helix with respect to every d_j [5].

Recently, M. Ghomi worked out the shadow problem given by H. Wente. And, He mentioned the shadow boundary in [8]. Ruiz-Hernández investigated that shadow boundaries are related to helix submanifolds in [12].

Helix hypersurfaces have been worked in nonflat ambient spaces in [6,7]. Cermelli and Di Scala have also studied helix hypersurfaces in liquid crystals in [2].

This paper is organized as follows. In section 2, we will give some basic properties in the general theory of weak r -helix, helix submanifolds and curves. And, in section 3, we will give the important relations between weak r -helix submanifolds and some special curves such as line of curvature, asymptotic curve and helix lines.

2. BASIC PROPERTIES

Definition 2.1. Given a submanifold $M \subset \mathbb{R}^n$ and an unitary vector d in \mathbb{R}^n , we say that M is a helix with respect to d if for each $q \in M$ the angle between d and $T_q M$ is constant.

Let us recall that a unitary vector d can be decomposed in its tangent and orthogonal components along the submanifold M , i.e. $d = \cos(\theta)T + \sin(\theta)\xi$ with $\|T\| = \|\xi\| = 1$, where $T \in TM$ and $\xi \in \mathfrak{O}(M)$. The angle between d and $T_q M$ is constant if and only if the tangential component of d has constant length $\|\cos(\theta)T\| = \cos(\theta)$. We can assume that $0 < \theta < \frac{\pi}{2}$ and we can say that M is a helix of angle θ .

We will call T and ξ the tangent and normal directions of the helix submanifold M . We can call d the helix direction of M and we will assume d always to be unitary [5].

Definition 2.2. Let $M \subset \mathbb{R}^n$ be a helix submanifold of angle $\theta \neq \frac{\pi}{2}$ w.r. to the direction $d \in \mathbb{R}^n$. We will call the integral curves of the tangent direction T of the helix M , the helix lines of M w.r.to d . Moreover, we say that a helix submanifold $M \subset \mathbb{R}^n$ is a ruled helix if all the helix lines of M are straight lines [5].

Proposition 2.1 The helix lines of a helix submanifold $M \subset \mathbb{R}^n$ are geodesic in M [5].

Definition 2.3. A submanifold $M \subset \mathbb{R}^n$ is a weak r -helix if there exist r linearly independent directions d_1, \dots, d_r , such that M is a helix with respect to every d_j [5].

Remark 2.1 We say that ξ is parallel normal in the direction $X \in TM$ if $\nabla_X^\perp \xi = 0$. Here, ∇^\perp denotes the normal connection of M induced by the standard covariant derivative of the Euclidean ambient. Let us denote by D the standard covariant derivative in \mathbb{R}^n and by ∇ the induced covariant derivative in M . Let A^ξ and V be the shape operator and the second fundamental form of $M \subset \mathbb{R}^n$ [5].

2000 *Mathematics Subject Classification.* 53A04, 53B25, 53C40, 53C50.

Key words and phrases. Weak r -helix submanifold; Line of curvature; Asymptotic curve; Helix line.

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Definition 2.4. Let M be a submanifold of the Riemannian manifold \mathbb{R}^n and let D be the Riemannian connexion on \mathbb{R}^n . For C^∞ fields X and Y with domain A on M (and tangent to M), define $\nabla_X Y$ and $V(X, Y)$ on A by decomposing $D_X Y$ into unique tangential and normal components, respectively; thus,

$$D_X Y = \nabla_X Y + V(X, Y).$$

Then, ∇ is the Riemannian connexion on M and V is a symmetric vector-valued 2-covariant C^∞ tensor called the second fundamental tensor. The above composition equation is called the Gauss equation [9].

Definition 2.5. Let M be a submanifold of the Riemannian manifold \mathbb{R}^n , let D be the Riemannian connexion on \mathbb{R}^n and let ∇ be the Riemannian connexion on M . Then, the formula of Weingarten

$$D_X \xi = -A^\xi(X) + \nabla_X^\perp \xi$$

for every X tangent to M and for every ξ normal to M . A^ξ is the shape operator associated to ξ also known as the Weingarten operator corresponding to ξ and ∇^\perp is the induced connexion in the normal bundle of M . $A^\xi(X)$ is also the tangent component of $-D_X \xi$ and will be denoted by $A^\xi(X) = \text{tang}(-D_X \xi)$ [10, 11].

Remark 2.2 Let us observe that for any helix euclidean submanifold M , the following system holds for every $X \in TM$, where the helix direction $d = \cos(\theta)T + \sin(\theta)\xi$.

$$\cos(\theta)\nabla_X T - \sin(\theta)A^\xi(X) = 0 \quad (2.1)$$

$$\cos(\theta)V(X, T) + \sin(\theta)\nabla_X^\perp \xi = 0 \quad (2.2)$$

[5].

Definition 2.6. If α is a (unit speed) curve in M with C^∞ unit tangent T , then $V(T, T)$ is called normal curvature vector field of α and $k_T = \|V(T, T)\|$ is called the normal curvature of α [9].

3. MAIN THEOREMS AND DEFINITIONS

Theorem 3.1. Let $M \subset \mathbb{R}^n$ be a weak r -helix submanifold with respect to the directions $d_j \in \mathbb{R}^n$, $j = 1, \dots, r$. Let D be Riemannian connexion (standard covariant derivative) on E^n and ∇ be Riemannian connexion on M . Let us assume that $\alpha : I \subset \mathbb{R} \rightarrow M$ is a unit speed (parametrized by arc length function s) curve on M with unit tangent T . Then, the normal component ξ_j of d_j is parallel normal in the direction T if and only if $T_j' \in TM$ along the curve α , where T_j is the unit tangent component of the direction d_j .

Proof. We assume that the normal component ξ_j of d_j is parallel normal in the direction T . Since T and $T_j \in TM$, from the Gauss equation in Definition (2.4),

$$D_T T_j = \nabla_T T_j + V(T, T_j) \quad (3.1)$$

According to the Theorem, since the normal component ξ_j of d_j is parallel normal in the direction T , i.e. $\nabla_T^\perp \xi_j = 0$ (see Remark 2.1), from (2.2) in Remark 2.2 ($0 < \theta < \frac{\pi}{2}$)

$$V(T, T_j) = 0 \quad (3.2)$$

So, by using (3.1), (3.2) and Frenet formulas, we have:

$$D_T T_j = \frac{dT_j}{ds} = T_j' = \nabla_T T_j.$$

That is, the vector field $T_j' \in T_{\alpha(t)}M$, where $T_{\alpha(t)}M$ is the tangent space of M .

Conversely, let us assume that $T_j' \in TM$ along the curve α . Then, from Gauss equation, $V(T, T_j) = 0$. Hence, from (2.2) in Remark 2.2 ($0 < \theta < \frac{\pi}{2}$), $\nabla_T^\perp \xi_j = 0$. That is, the normal component ξ_j of d_j is parallel normal in the direction T . This completes the proof. \square

Theorem 3.2. Let $M \subset \mathbb{R}^n$ be a weak r -helix submanifold with respect to the directions $d_j \in \mathbb{R}^n$, $j = 1, \dots, r$. Let D be Riemannian connexion (standard covariant derivative) on E^n and ∇ be Riemannian connexion on M . Then, the normal curvature of the unit integral curve α_j of T_j equals $|k_j|$, where T_j is the unit tangent component of the direction d_j and k_j is the first curvature of α_j .

Proof. Since α_j is the integral curve of $T_j \in TM$, we can write

$$\frac{d\alpha_j}{ds} = T_j$$

along the curve α_j . Also, since $T_j \in TM$, from the Gauss equation in Definition (2.4)

$$D_{T_j}T_j = \nabla_{T_j}T_j + V(T_j, T_j). \quad (3.3)$$

On the other hand, the integral curves of T_j are geodesics in M according to the Proposition 2.1. That is

$$\nabla_{T_j}T_j = 0. \quad (3.4)$$

So, by using (3.3),(3.4) and Frenet formulas, we have:

$$\begin{aligned} D_{T_j}T_j &= \frac{dT_j}{ds} = T_j' \\ &= k_j V_{2_j} \quad (V_{2_j} \text{ is the unit principal normal of } \alpha_j) \\ &= V(T_j, T_j). \end{aligned}$$

Hence, we get $\|V(T_j, T_j)\| = \|k_j V_{2_j}\| = |k_j|$. This completes the proof. \square

Lemma 3.1. *Let $M \subset \mathbb{R}^n$ be a weak r -helix submanifold with respect to the directions $d_j \in \mathbb{R}^n$, $j = 1, \dots, r$. Let D be the Riemannian connexion on \mathbb{R}^n and let ∇ be the Riemannian connexion on M . If $\alpha_j : I \subset \mathbb{R} \rightarrow M$ is the (unit speed) integral curve of the unit tangent direction T_j of d_j , then $V_{2_j} \in \vartheta(M)$, where V_{2_j} is the unit principal normal of α_j and $\vartheta(M)$ is the normal space of M .*

Proof. Since $T_j \in TM$, from the Gauss equation in Definition 2.4,

$$D_{T_j}T_j = \nabla_{T_j}T_j + V(T_j, T_j) \quad (3.5)$$

According to the Proposition 2.1, since α_j is a geodesic curve on M ,

$$\nabla_{T_j}T_j = 0 \quad (3.6)$$

So, by using (3.5), (3.6) and Frenet formulas, we get:

$$D_{T_j}T_j = k_j V_{2_j} = V(T_j, T_j). \quad (k_j \text{ is the first curvature of } \alpha_j)$$

That is, the vector field $V_{2_j} \in \vartheta(M)$ along the curve α_j . This completes the proof. \square

Definition 3.1. *Let M be a submanifold of the Riemannian manifold of \mathbb{R}^n and let A^ξ be the shape operator in a direction $\xi \in \vartheta(M)$. For a vector field $X \in TM$, if $\langle A^\xi(X), X \rangle = 0$, then X will be called asymptotic in the direction ξ .*

Definition 3.2. *Let M be a submanifold of the Riemannian manifold of \mathbb{R}^n and let $\alpha : I \subset \mathbb{R} \rightarrow M$ be a unit speed curve in M . If $\langle A^\xi(T), T \rangle = 0$, then the curve α will be called an asymptotic curve in the direction ξ , where T unit tangent vector field of α and $\xi \in \vartheta(M)$ (normal space).*

Definition 3.3. *The second normal space of $M \subset \mathbb{R}^n$ consist of the normal vectors, $\xi \in \vartheta(M)$, such that the shape operator in its direction is zero, i.e. $A^\xi = 0$ [5].*

Theorem 3.3. *Let M be a submanifold of the Riemannian manifold of \mathbb{R}^n . Then every curve in M is asymptotic in the direction ξ if $\xi \in \vartheta(M)$ is an element of the second normal space $M \subset \mathbb{R}^n$.*

Proof. Let assume that $\xi \in \vartheta(M)$ is an element of the second normal space $M \subset \mathbb{R}^n$ and let α be an arbitrary curve with the unit tangent T in M . Then, from Definition 3.3, $A^\xi = 0$ for every $X \in TM$. And, in particular since $T \in TM$, $A^\xi(T) = 0$. Hence, we deduce that $\langle A^\xi(T), T \rangle = 0$. Consequently, since α is an arbitrary curve, from Definition 3.2, every curve in M is asymptotic in the direction ξ . This completes the proof. \square

Theorem 3.4. *Let $M \subset \mathbb{R}^n$ be a weak r -helix submanifold with respect to the directions $d_j \in \mathbb{R}^n$, $j = 1, \dots, r$. Let D be the Riemannian connexion on \mathbb{R}^n and let ∇ be the Riemannian connexion on M . If T_j is parallel in M , i.e. $\nabla_X T_j = 0$ for every $X \in TM$, then every $X \in TM$ is asymptotic in the direction ξ_j . Here, T_j is tangent component of d_j and ξ_j is normal component of d_j . Conversely, if every $X \in TM$ is asymptotic in the direction ξ_j , then T_j is parallel in M .*

Proof. We assume that T_j is parallel in M . That is, $\nabla_X T_j = 0$ for every $X \in TM$. So, by using the equation (2.1), we deduce that $A^{\xi_j}(X) = 0$ for every $X \in TM$ ($\theta \neq 0$). Due to the fact that $A^{\xi_j}(X) = 0$ for every $X \in TM$, $\langle A^{\xi_j}(X), X \rangle = 0$ for every $X \in TM$. Consequently, by using the Definition 3.1, every $X \in TM$ is asymptotic in the direction ξ_j .

Conversely, let us assume that every $X \in TM$ is asymptotic in the direction ξ_j . Then, $\langle A^{\xi_j}(X), X \rangle = 0$ for every $X \in TM$. Moreover, by using the equation (2.1), we obtain $\langle \nabla_X T_j, X \rangle = 0$ for every $X \in TM$ ($\theta \neq \frac{\pi}{2}$). Hence, we have $\nabla_X T_j = 0$. This completes the proof. \square

Theorem 3.5. *Let $M \subset \mathbb{R}^n$ be a weak r -helix submanifold with respect to the directions $d_j \in \mathbb{R}^n$, $j = 1, \dots, r$ and let D be the Riemannian connexion on \mathbb{R}^n . If $\alpha_j : I \subset \mathbb{R} \rightarrow M$ is the (unit speed) integral curve of the unit tangent direction T_j of d_j , then the curve α_j is asymptotic in the direction ξ_j , where ξ_j is orthogonal component of d_j .*

Proof. Since M is a weak r -helix submanifold, we can decompose the direction d_j in its tangent and normal components:

$$d_j = \cos(\theta_j)T_j + \sin(\theta_j)\xi_j \quad (3.7)$$

From (3.7), by taking derivatives on both sides along the curve α_j , we have:

$$0 = \cos(\theta_j)T_j' + \sin(\theta_j)\xi_j'$$

and by using Frenet formulas, we get:

$$0 = (k_j \cos(\theta_j))V_{2_j} + \sin(\theta_j)\xi_j' \quad (3.8)$$

According to the Lemma 3.1, since $V_{2_j} \in \vartheta(M)$ along the curve α_j , from (3.8), we deduce that $\xi_j' \in \vartheta(M)$. On the other hand,

$$\begin{aligned} A^{\xi_j}(T_j) &= \text{tang}(-D_{T_j}\xi_j) \\ &= \text{tang}(-\xi_j') \end{aligned}$$

and since $\xi_j' \in \vartheta(M)$, we obtain $\text{tang}(-\xi_j') = 0$. So $\langle A^{\xi_j}(T_j), T_j \rangle = 0$. This completes the proof. \square

Definition 3.4. *Given an Euclidean submanifold of arbitrary codimension $M \subset \mathbb{R}^n$. A curve α in M is called a line of curvature if its tangent T is a principal vector at each of its points. In other words, when T (the tangent of α) is a principal vector at each of its points, for an arbitrary normal vector field $\xi \in \vartheta(M)$, the shape operator A^ξ associated to ξ says $A^\xi(T) = \text{tang}(-D_T\xi) = \lambda_j T$ along the curve α , where λ_j is a principal curvature and D be the Riemannian connexion (standard covariant derivative) on \mathbb{R}^n [1].*

Theorem 3.6. *Let $M \subset \mathbb{R}^n$ be a weak r -helix submanifold with respect to the directions $d_j \in \mathbb{R}^n$, $j = 1, \dots, r$ and let D be the Riemannian connexion on \mathbb{R}^n . Let us assume that $\alpha : I \subset \mathbb{R} \rightarrow M$ is a (unit speed) line of curvature (not a straight line) for a normal vector field $\xi \in \vartheta(M)$, where ξ' is TM along the curve α . Then, $d_j \notin \text{Sp}\{\xi, T\}$ along the curve α for all the directions d_j , where T is the unit tangent vector field of α .*

Proof. We assume that $d_j \in \text{Sp}\{\xi, T\}$ along the curve α for any direction d_j . Since M is a weak r -helix submanifold, we can decompose the direction d_j in its tangent and normal components:

$$d_j = \cos(\theta_j)\xi + \sin(\theta_j)T, \quad (3.9)$$

where θ_j is constant. From (3.9), by taking derivatives on both sides along the curve α , we get:

$$0 = \cos(\theta_j)\xi' + \sin(\theta_j)T' \quad (3.10)$$

Moreover, since α is a line of curvature (not a straight line) for a normal vector field $\xi \in \vartheta(M)$,

$$A^\xi(T) = \text{tang}(-D_T\xi) = \text{tang}(-\xi') = \lambda_j T$$

along the curve α . According to the Theorem, since ξ' is TM along the curve α ,

$$\text{tang}(-\xi') = -\xi' = \lambda_j T \quad (3.11)$$

By using the equations (3.10) and (3.11), we deduce that the system $\{T, T'\}$ is linear dependent. But, the system $\{T, T'\}$ is never linear dependent. This is a contradiction. This completes the proof. \square

This latter Theorem has the following corollary.

Corollary 3.1. *For an arbitrary direction $d_j \in \mathbb{R}^n$, if $d_j \in Sp\{\xi, T\}$, then the curve α is not a line of curvature with respect to $\xi \in \vartheta(M)$, where T is the unit tangent vector field of α .*

Theorem 3.7. *Let $M \subset \mathbb{R}^n$ be a full submanifold (it is not contained in a hyperplane of the ambient \mathbb{R}^n) which is a helix with respect to the direction d . Let ξ be the normal component of d , i.e. $d = \cos(\theta)T + \sin(\theta)\xi$. Then M is a ruled helix (see definition 2.2) if and only if ξ is $\nabla_T^\perp \xi = 0$ [5].*

Theorem 3.8. *Let $M \subset \mathbb{R}^n$ be a full submanifold which is a helix with respect to the direction $d = \cos(\theta)T + \sin(\theta)\xi$ ($\theta \neq 0$). Then, M is a ruled helix if and only if the normal curvatures of the helix lines of M with respect to d equal zero.*

Proof. We assume that M is a ruled helix. Then, from Theorem 3.7, $\nabla_T^\perp \xi = 0$ for the direction d . So, from (2.2) in Remark 2.2, $V(T, T) = 0$ (since M is full or $\theta \neq \frac{\pi}{2}$). Hence, the normal curvatures of the helix lines of M with respect to d equal zero.

Conversely, let us assume that the normal curvatures of the helix lines of M with respect to d equal zero. In other words, $V(T, T) = 0$. Since $\theta \neq 0$, from (2.2) in Remark 2.2, we obtain $\nabla_T^\perp \xi = 0$. And, from Theorem 3.7, M is a ruled helix. This completes the proof. \square

This latter Theorem has the following corollary.

Corollary 3.2. *In Theorem 3.8, in particular, let us assume that M is a hypersurface in \mathbb{R}^n . Then, since the helix lines of M are straight lines of \mathbb{R}^n (see Lemma 2.5 in [4]), M is always a ruled helix. Finally, the normal curvatures of the helix lines of M with respect to d equal always zero.*

Acknowledgment. The authors would like to thank referees for their valuable suggestions and comments that helped to improve the presentation of this paper.

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